Infinite log-concavity: developments and conjectures

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Abstract

Given a sequence $(a_k) = a_0, a_1, a_2, \ldots$ of real numbers, define a new sequence $\mathcal{L}(a_k) = (b_k)$ where $b_k = a_k^2 - a_{k-1}a_{k+1}$. So (a_k) is log-concave if and only if (b_k) is a nonnegative sequence. Call (a_k) *infinitely log-concave* if $\mathcal{L}^i(a_k)$ is nonnegative for all $i \geq 1$. Boros and Moll [4] conjectured that the rows of Pascal's triangle are infinitely log-concave. Using a computer and a stronger version of log-concavity, we prove their conjecture for the nth row for all $n \leq 1450$. We also use our methods to give a simple proof of a recent result of Uminsky and Yeats [30] about regions of infinite log-concavity. We investigate related questions about the columns of Pascal's triangle, q-analogues, symmetric functions, real-rooted polynomials, and Toeplitz matrices. In addition, we offer several conjectures.

1 Introduction

Let

$$(a_k) = (a_k)_{k\geq 0} = a_0, a_1, a_2, \ldots$$

be a sequence of real numbers. It will be convenient to extend the sequence to negative indices by letting $a_k = 0$ for $k < 0$. Also, if $(a_k) = a_0, a_1, \ldots, a_n$ is a finite sequence then we let $a_k = 0$ for $k > n$.

Define the $\mathcal{L}$*-operator* on sequences to be $\mathcal{L}(a_k) = (b_k)$ where $b_k = a_k^2 - a_{k-1}a_{k+1}$. Call a sequence i*-fold log-concave* if $\mathcal{L}^i(a_k)$ is a nonnegative sequence. So log-concavity in the ordinary sense is 1-fold log-concavity. Log-concave sequences arise in many areas of algebra, combinatorics, and geometry. See the survey articles of Stanley [25] and Brenti [8] for more information.

Boros and Moll [4, page 157] defined (a_k) to be *infinitely log-concave* if it is i-fold log-concave for all $i \geq 1$. They introduced this definition in conjunction with the study of a specialization of the Jacobi polynomials whose coefficient sequence they conjectured to be infinitely log-concave. Kauers and Paule [16] used a computer algebra package to prove this conjecture for ordinary log-concavity. Since the coefficients of these polynomials can be expressed in terms of binomial coefficients, Boros and Moll also made the statement:

"Prove that the binomial coefficients are ∞-logconcave."

We will take this to be a conjecture that the rows of Pascal's triangle are infinitely log-concave, although we will later discuss the columns and other lines. When given a function of more than one variable, we will subscript the $\mathcal{L}$-operator by the parameter which is varying to form the sequence. So $\mathcal{L}_k\binom{n}{k}$ would refer to the operator acting on the sequence $\binom{n}{k}_{k\geq 0}$. Note that we drop the sequence parentheses for sequences of binomial coefficients to improve readability. We now restate the Boros-Moll conjecture formally.

Conjecture 1.1. *The sequence* $\binom{n}{k}_{k\geq 0}$ *is infinitely log-concave for all* $n \geq 0$.

In the next section, we use a strengthened version of log-concavity and computer calculations to verify Conjecture 1.1 for all $n \leq 1450$. Uminsky and Yeats [30] set up a correspondence between certain symmetric sequences and points in $\mathbb{R}^m$. They then described an infinite region $\mathcal{R} \subset \mathbb{R}^m$ bounded by hypersurfaces and such that each sequence corresponding to a point of $\mathcal{R}$ is infinitely log-concave. In Section 3, we show how our methods can be used to give a simple derivation of one of their main theorems. We investigate infinite log-concavity of the columns and other lines of Pascal's triangle in Section 4. Section 5 is devoted to two q-analogues of the binomial coefficients. For the Gaussian polynomials, we show that certain analogues of some infinite log-concavity conjectures are false while others appear to be true. In contrast, our second q-analogue seems to retain all the log-concavity properties of the binomial coefficients. In Section 6, after showing why the sequence $(h_k)_{k\geq 0}$ of complete homogeneous symmetric is an appropriate analogue of sequences of

binomial coefficients, we explore its log-concavity properties. We end with a section of related results and questions about real-rooted polynomials and Toeplitz matrices.

While one purpose of this article is to present our results, we have written it with two more targets in mind. The first is to convince our audience that infinite log-concavity is a fundamental concept. We hope that its definition as a natural extension of traditional log-concavity helps to make this case. Our second aspiration is to attract others to work on the subject; to that end, we have presented several open problems. These conjectures each represent fundamental questions in the area, so even solutions of special cases may be interesting.

2 Rows of Pascal's triangle

One of the difficulties with proving the Boros-Moll conjecture is that log-concavity is not preserved by the $\mathcal{L}$-operator. For example, the sequence $4, 5, 4$ is log-concave but $\mathcal{L}(4, 5, 4) = 16, 9, 16$ is not. So we will seek a condition stronger than log-concavity which is preserved by $\mathcal{L}$. Given $r \in \mathbb{R}$, we say that a sequence (a_k) is *r-factor log-concave* if

$$a_k^2 \geq r a_{k-1} a_{k+1} \tag{1}$$

for all k. Clearly this implies log-concavity if $r \geq 1$.

We seek an $r > 1$ such that (a_k) being r-factor log-concave implies that $(b_k) = \mathcal{L}(a_k)$ is as well. Assume the original sequence is nonnegative. Then expanding $r b_{k-1} b_{k+1} \leq b_k^2$ in terms of the a_k and rearranging the summands, we see that this is equivalent to proving

$$(r-1)a_{k-1}^2 a_{k+1}^2 + 2a_{k-1}a_k^2 a_{k+1} \leq a_k^4 + r a_{k-2} a_k (a_{k+1}^2 - a_k a_{k+2}) + r a_{k-1}^2 a_k a_{k+2}.$$

By our assumptions, the two expressions with factors of r on the right are nonnegative, so it suffices to prove the inequality obtained when these are dropped. Applying (1) to the left-hand side gives

$$(r-1)a_{k-1}^2 a_{k+1}^2 + 2a_{k-1}a_k^2 a_{k+1} \leq \frac{r-1}{r^2} a_k^4 + \frac{2}{r} a_k^4.$$

So we will be done if

$$\frac{r-1}{r^2} + \frac{2}{r} = 1.$$

Finding the root $r_0 > 1$ of the corresponding quadratic equation finishes the proof of the first assertion of the following lemma, while the second assertion follows easily from the first.

Lemma 2.1. *Let (a_k) be a nonnegative sequence and let $r_0 = (3+\sqrt{5})/2$. Then (a_k) being r_0-factor log-concave implies that $\mathcal{L}(a_k)$ is too. So in this case (a_k) is infinitely log-concave.* $\square$

Now to prove that any row of Pascal's triangle is infinitely log-concave, one merely lets a computer find $\mathcal{L}^i_k\binom{n}{k}$ for i up to some bound I. If these sequences are all nonnegative and $\mathcal{L}^I_k\binom{n}{k}$ is r_0-factor log-concave, then the previous lemma shows that this row is infinitely log-concave. Using this technique, we have obtained the following theorem.

Theorem 2.2. *The sequence $\binom{n}{k}_{k\geq 0}$ is infinitely log-concave for all $n \leq 1450$.* $\square$

We note that the necessary value of I increases slowly with increasing n. As an example, when $n = 100$, our technique works with $I = 5$, while for $n = 1000$, we need $I = 8$.

Of course, the method developed in this section can be applied to any sequence such that $\mathcal{L}^i(a_k)$ is r_0-factor log-concave for some i. In particular, it is interesting to try it on the original sequence which motivated Boros and Moll [4] to define infinite log-concavity. They were studying the polynomial

$$P_m(x) = \sum_{\ell=0}^{m} d_\ell(m)x^\ell \tag{2}$$

where

$$d_\ell(m) = \sum_{j=\ell}^{m} 2^{j-2m}\binom{2m-2j}{m-j}\binom{m+j}{m}\binom{j}{\ell}.$$

Kauers [private communication] has used our method to verify infinite log-concavity of the sequence $(d_\ell(m))_{\ell\geq 0}$ for $m \leq 129$. For such values of m, $\mathcal{L}^5_\ell$ applied to the sequence is r_0-factor log-concave.

3 A region of infinite log-concavity

Uminsky and Yeats [30] took a different approach to the Boros-Moll Conjecture as described in the Introduction. Since they were motivated by the rows of Pascal's triangle, they only considered real sequences $a_0, a_1, \ldots, a_n$ which are symmetric (in that $a_k = a_{n-k}$ for all k) and satisfy $a_0 = a_n = 1$. Each such sequence corresponds to a point $(a_1, \ldots, a_m) \in \mathbb{R}^m$ where $m = \lfloor n/2 \rfloor$.

Their region, $\mathcal{R}$, whose points all correspond to infinitely log-concave sequences, is bounded by m parametrically defined hypersurfaces. The parameters are x and

$d_1, d_2, \ldots, d_m$ and it will be convenient to have the notation

$$s_k = \sum_{i=1}^{k} d_i.$$

We will also need $r_1 = (1+\sqrt{5})/2$. Note that $r_1^2 = r_0$. The kth hypersurface, $1 \le k < m$, is defined as

$$\mathcal{H}_k = \{(x^{s_1}, \ldots, x^{s_{k-1}}, r_1 x^{s_k}, x^{s_{k+1}+d_k-d_{k+1}}, \ldots, x^{s_m+d_k-d_{k+1}}) : \\ x \ge 1, \quad 1 = d_1 > \cdots > d_k > d_{k+2} > \cdots > d_m > 0\},$$

while

$$\mathcal{H}_m = \{(x^{s_1}, \ldots, x^{s_{m-1}}, c x^{s_{m-1}}) : \ x \ge 1, \quad 1 = d_1 > \cdots > d_{m-1} > 0\},$$

where

$$c = \begin{cases} r_1 & \text{if } n = 2m, \\ 2 & \text{if } n = 2m+1. \end{cases}$$

Let us say that the *correct side* of $\mathcal{H}_k$ for $1 \le k \le m$ consists of those points in $\mathbb{R}^m$ that can be obtained from a point on $\mathcal{H}_k$ by increasing the kth coordinate. Then let $\mathcal{R}$ be the region of all points in $\mathbb{R}^m$ having increasing coordinates and lying on the correct side of $\mathcal{H}_k$ for all k. We will show how our method of the previous section can be used to give a simple proof of one of Uminsky and Yeats' main theorems. But first we need a modified version of Lemma 2.1 to take care of the case when $n = 2m+1$.

Lemma 3.1. *Let $a_0, a_1, \ldots, a_{2m+1}$ be a symmetric, nonnegative sequence such that*

(i) $a_k^2 \ge r_0 a_{k-1} a_{k+1}$ *for* $k < m$, *and*

(ii) $a_m \ge 2a_{m-1}$.

Then $\mathcal{L}(a_k)$ has the same properties, which implies that (a_k) is infinitely log-concave.

Proof. Clearly $\mathcal{L}(a_k)$ is still symmetric. To show that the other two properties persist, note that in demonstrating Lemma 2.1 we actually proved more. In particular, we showed that if equation (1) holds at index k of the sequence (a_k) (with $r = r_0$), then it also holds at index k of the sequence $\mathcal{L}(a_k)$ provided that the original sequence is log-concave. Note that the assumptions of the current lemma imply log-concavity of (a_k): This is clear at indices $k \ne m, m+1$ because of condition (i). Also, using symmetry and multiplying condition (ii) by a_m gives $a_m^2 \ge 2a_{m-1}a_m = 2a_{m-1}a_{m+1}$ (and symmetrically for $k = m+1$).

So now we know that condition (i) is also true for $\mathcal{L}(a_k)$. As for condition (ii), using symmetry we see that we need to prove

$$a_m^2 - a_{m-1}a_m \geq 2\left(a_{m-1}^2 - a_{m-2}a_m\right).$$

Rearranging terms and dropping one of them shows that it suffices to demonstrate

$$2a_{m-1}^2 + a_{m-1}a_m \leq a_m^2.$$

But this is true because of (ii), and we are done. □

Theorem 3.2 ([30]). *Any sequence corresponding to a point of $\mathcal{R}$ is infinitely log-concave.*

Proof. It suffices to show that the sequence satisfies the hypotheses of Lemma 2.1 when $n = 2m$, or Lemma 3.1 when $n = 2m+1$.

Suppose first that $k < m$. Being on the correct side of $\mathcal{H}_k$ is equivalent to there being values of the parameters such that

$$a_k^2 \;\geq\; (r_1 x^{s_k})^2 \;=\; r_1^2 x^{(s_{k-1}+d_k)+(s_{k+1}-d_{k+1})} \;=\; r_0 a_{k-1}a_{k+1}.$$

Thus we have the necessary inequalities for this range of k.

If $k = m$ then we can use an argument as in the previous paragraph if $n = 2m$. If $n = 2m+1$, then being on the correct side of $\mathcal{H}_m$ is equivalent to

$$a_m \geq 2x^{s_{m-1}} = 2a_{m-1}.$$

This is precisely condition (ii) of Lemma 3.1, which finishes the proof. □

4 Columns and other lines of Pascal's triangle

While we have treated Boros and Moll's statement about the infinite log-concavity of the binomial coefficients to be a statement about the rows of Pascal's triangle, their wording also suggests an examination of the columns.

Conjecture 4.1. *The sequence $\binom{n}{k}_{n\geq k}$ is infinitely log-concave for all fixed $k \geq 0$.*

We will give two pieces of evidence for this conjecture. One is a demonstration that various columns corresponding to small values of k are infinitely log-concave. Another is a proof that $\mathcal{L}_n^i\binom{n}{k}$ is nonnegative for certain values of i and all k.

Proposition 4.2. *The sequence $\binom{n}{k}_{n\geq k}$ is infinitely log-concave for $0 \leq k \leq 2$.*

Proof. When $k = 0$ we have, for all $i \geq 1$,

$$\mathcal{L}_n^i \binom{n}{0} = (1, 0, 0, 0, \ldots).$$

For $k = 1$ we obtain

$$\mathcal{L}_n \binom{n}{1} = (1, 1, 1, \ldots)$$

so infinite log-concavity follows from the $k = 0$ case. The sequence when $k = 2$ is a fixed point of the $\mathcal{L}$-operator, again implying infinite log-concavity. □

In what follows, we use the notation $L(a_k)$ for the kth element of the sequence $\mathcal{L}(a_k)$, and similarly for L_k and L_n.

Proposition 4.3. *The sequence $\mathcal{L}_n^i \binom{n}{k}$ is nonnegative for all k and for $0 \leq i \leq 4$.*

Proof. By the previous proposition, we only need to check $k \geq 3$. Using the expression for a binomial coefficient in terms of factorials, it is easy to derive the following expressions:

$$L_n \binom{n}{k} = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$$

and

$$L_n^2 \binom{n}{k} = \frac{2}{n^2(n-1)} \binom{n}{k}^2 \binom{n}{k-1} \binom{n}{k-2}.$$

With a little more work, one can show that $L_n^3 \binom{n}{k}$ can be expressed as a product of nonnegative factors times the polynomial

$$(4k - 6)n^2 - (4k^2 - 10k + 6)n - k^2.$$

To show that this is nonnegative, we write $n = k + m$ for $m \geq 0$ to get

$$(4k - 6)m^2 + (4k^2 - 2k - 6)m + (3k^2 - 6k).$$

But the coefficients of the powers of m are all positive for $k \geq 3$, so we are done with the case $i = 3$.

When $i = 4$, we follow the same procedure, only now the polynomial in m has coefficients which are polynomials in k up to degree 7. For example, the coefficient of m^3 is

$$528k^7 - 8k^6 - 11,248k^5 + 25,360k^4 - 5,888k^3 - 24,296k^2 + 16,080k - 1,584.$$

To make sure this is nonnegative for integral $k \geq 3$, one rewrites the polynomial as

$$(528k^2 - 8k - 11,248)k^5 + (25,360k^2 - 5,888k - 24,296)k^2 + (16,080k - 1,584),$$

finds the smallest k such that each of the factors in parentheses is nonnegative from this value on, and then checks any remaining k by direct substitution. □

Kauers and Paule [16] proved that the rows of Pascal's triangle are i-fold log-concave for $i \leq 5$. Kauers [private communication] has used their techniques to confirm Proposition 4.3 and to also check the case $i = 5$ for the columns. For the latter case, Kauers used a computer to determine

$$\frac{(\mathcal{L}_n^5 \binom{n}{k})}{\binom{n}{k}^{2^5}} \tag{3}$$

explicitly, which is just a rational function in n and k. He then showed that (3) is nonnegative by means of cylindrical algebraic decomposition. We refer the interested reader to [16] and the references therein for more information on such techniques.

More generally, we can look at an arbitrary line in Pascal's triangle, i.e., consider the sequence $\binom{n+mu}{k+mv}_{m\geq 0}$. The unimodality and (1-fold) log-concavity of such sequences has been investigated in [3, 27, 28, 29]. We do not require that u and v be coprime, so such sequences need not contain all of the binomial coefficients in which a geometric line would intersect Pascal's triangle, e.g., a sequence such as $\binom{n}{0}, \binom{n}{2}, \binom{n}{4}, \ldots$ would be included. By letting $u < 0$, one can get a finite truncation of a column. For example, if $n = 5$, $k = 3$, $u = -1$, and $v = 0$ then we get the sequence

$$\binom{5}{3}, \binom{4}{3}, \binom{3}{3}$$

which is not even 2-fold log-concave. So we will only consider $u \geq 0$. Also

$$\binom{n+mu}{k+mv} = \binom{n+mu}{n-k+m(u-v)}$$

so we can also assume $v \geq 0$.

We offer the following conjecture, which includes Conjecture 1.1 as a special case.

Conjecture 4.4. *Suppose that u and v are distinct nonnegative integers. Then $\binom{n+mu}{mv}_{m\geq 0}$ is infinitely log-concave for all $n \geq 0$ if and only if $u < v$ or $v = 0$.*

We first give a quick proof of the "only if" direction. Supposing that $u > v \geq 1$, we consider the sequence

$$\binom{0}{0}, \binom{u}{v}, \binom{2u}{2v}, \ldots$$

obtained when $n = 0$. We claim that this sequence is not even log-concave and that log-concavity fails at the second term. Indeed, the fact that $\binom{u}{v}^2 < \binom{2u}{2v}$ follows immediately from the identity

$$\binom{u}{0}\binom{u}{2v} + \binom{u}{1}\binom{u}{2v-1} + \cdots + \binom{u}{v}\binom{u}{v} + \cdots + \binom{u}{2v}\binom{u}{0} = \binom{2u}{2v},$$

which is a special case of Vandermonde's Convolution.

The proof just given shows that subsequences of the columns of Pascal's triangle are the only *infinite* sequences of the form $\binom{n+mu}{mv}_{m\geq 0}$ that can possibly be infinitely log-concave. We also note that the previous conjecture says nothing about what happens on the diagonal $u = v$. Of course, the case $u = v = 1$ is Conjecture 4.1. For other diagonal values, the evidence is conflicting. One can show by computer that $\binom{n+mu}{mu}_{m\geq 0}$ is not 4-fold log-concave for $n = 2$ and any $2 \leq u \leq 500$. However, this is the only *known* value of n for which $\binom{n+mu}{mu}_{m\geq 0}$ is not an infinitely log-concave sequence for some $u \geq 1$.

We conclude this section by offering considerable computational evidence in favor of the "if" direction of Conjecture 4.4. Theorem 2.2 provides such evidence when $u = 0$ and $v = 1$. Since all other sequences with $u < v$ have a finite number of nonzero entries, we can use the r_0-factor log-concavity technique for these sequences as well. For all $n \leq 500$, $2 \leq v \leq 20$ and $0 \leq u < v$, we have checked that $\binom{n+mu}{mv}_{m\geq 0}$ is infinitely log-concave.

5 q-analogues

This section will be devoted to discussing two q-analogues of binomial coefficients. For the Gaussian polynomials, we will see that the corresponding generalization of Conjecture 1.1 is false, and we show one exact reason why it fails. In contrast, the corresponding generalization of Conjecture 4.1 appears to be true. This shows how delicate these conjectures are and may in part explain why they seem to be difficult to prove. After introducing our second q-analogue, we conjecture that the corresponding generalizations of Conjectures 1.1, 4.1 and 4.4 are all true. This second q-analogue arises in the study of quantum groups; see, for example, the books of Jantzen [15] and Majid [21].

Let q be a variable and consider a polynomial $f(q) \in \mathbb{R}[q]$. Call $f(q)$ *q-nonnegative* if all the coefficients of $f(q)$ are nonnegative. Apply the $\mathcal{L}$-operator to sequences of polynomials $(f_k(q))$ in the obvious way. Call such a sequence *q-log-concave* if $\mathcal{L}(f_k(q))$ is a sequence of q-nonnegative polynomials, with i-fold q-log-concavity and infinite q-log-concavity defined similarly.

We will be particularly interested in the Gaussian polynomials. The standard *q-analogue of the nonnegative integer* n is

$$[n] = [n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}.$$

Then, for $0 \leq k \leq n$, the *Gaussian polynomials* or *q-binomial coefficients* are defined

as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

where $[n]_q! = [1]_q[2]_q \cdots [n]_q$. For more information, including proofs of the assertions made in the next paragraph, see the book of Andrews [2].

Clearly substituting $q = 1$ gives $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_1 = \binom{n}{k}$. Also, it is well known that the Gaussian polynomials are indeed q-nonnegative polynomials. In fact, they have various combinatorial interpretations, one of which we will need. An (integer) *partition of* n is a weakly decreasing positive integer sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $|\lambda| \stackrel{\text{def}}{=} \sum_i \lambda_i = n$. The λ_i are called *parts*. For notational convenience, if a part k is repeated r times in a partition λ then we will denote this by writing k^r in the sequence for λ. We say that λ *fits inside an* $s \times t$ *box* if $\lambda_1 \le t$ and $\ell \le s$. Denote the set of all such partitions by $P(s,t)$. It is well known, and easy to prove by induction on n, that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{\lambda \in P(n-k,k)} q^{|\lambda|}. \tag{4}$$

We are almost ready to prove that the sequence $\left(\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]\right)_{k \ge 0}$ is not infinitely q-log-concave. In fact, we will show it is not even 2-fold q-log-concave. First we need a lemma. In it, we use $\operatorname{mint} f(q)$ to denote the nonzero term of least degree in $f(q)$.

Lemma 5.1. *Let* $L_k\left(\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]\right) = B_k(q)$. *Then for* $k \le n/2$,

$$\operatorname{mint} B_k(q) = \begin{cases} q^k & \text{if } k < n/2, \\ 2q^k & \text{if } k = n/2. \end{cases}$$

Proof. Since $B_k(q) = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]^2 - \left[\begin{smallmatrix} n \\ k-1 \end{smallmatrix}\right]\left[\begin{smallmatrix} n \\ k+1 \end{smallmatrix}\right]$ it suffices to prove, in view of (4), the following two statements. If $i \le k$ and

$$(\lambda, \mu) \in P(n-k+1, k-1) \times P(n-k-1, k+1)$$

with $|\lambda| + |\mu| = i$, then $(\lambda, \mu) \in P(n-k,k)^2$. Furthermore, the number of elements in $P(n-k,k)^2 - P(n-k+1,k-1) \times P(n-k-1,k+1)$ is 0 or 1 or 2 depending on whether $i < k$ or $i = k < n/2$ or $i = k = n/2$, respectively.

The first statement is an easy consequence of $|\lambda| + |\mu| = i \le k \le n-k$. A similar argument works for the $i < k$ case of the second statement. If $i = k$ then the pair $((k), \emptyset)$ is in the difference and if $i = k = n/2$ then the pair $(\emptyset, (1^k))$ is as well. $\square$

Proposition 5.2. *Let* $L_k^2\left(\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]\right) = C_k(q)$. *Then for* $n \ge 2$ *and* $k = \lfloor n/2 \rfloor$ *we have*

$$\operatorname{mint} C_k(q) = -q^{n-2}.$$

Consequently, $\left(\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]\right)_{k \ge 0}$ *is not 2-fold q-log-concave.*

Proof. The proofs for n even and odd are similar, so we will only do the former. So suppose $n = 2k$ and consider

$$C_k(q) = B_k(q)^2 - B_{k-1}(q)B_{k+1}(q) = B_k(q)^2 - B_{k-1}(q)^2.$$

By the previous lemma $\operatorname{mint} B_k(q)^2 = 4q^{2k}$ and $\operatorname{mint} B_{k-1}(q)^2 = q^{2k-2}$. Thus $\operatorname{mint} C_k(q) = -q^{2k-2} = -q^{n-2}$ as desired. $\square$

After what we have just proved, it may seem surprising that the following conjecture, which is a q-analogue of Conjecture 4.1, does seem to hold.

Conjecture 5.3. *The sequence $\left(\begin{bmatrix} n \\ k \end{bmatrix}\right)_{n \geq k}$ is infinitely q-log-concave for all fixed $k \geq 0$.*

As evidence, we will prove a q-analogue of Proposition 4.2 and comment on Proposition 4.3 in this setting.

Proposition 5.4. *The sequence $\left(\begin{bmatrix} n \\ k \end{bmatrix}\right)_{n \geq k}$ is infinitely q-log-concave for $0 \leq k \leq 2$.*

Proof. When $k = 0$ one has the same sequence as when $q = 1$.

When $k = 1$ we claim that

$$\mathcal{L}\left(\begin{bmatrix} n \\ 1 \end{bmatrix}\right) = (1, q, q^2, q^3, \ldots).$$

Indeed,

$$\begin{aligned} [n]^2 - [n-1][n+1] &= \frac{(1-q^n)^2 - (1-q^{n-1})(1-q^{n+1})}{(1-q)^2} \\ &= \frac{q^{n-1} - 2q^n + q^{n+1}}{(1-q)^2} \\ &= q^{n-1} \end{aligned}$$

(and recall that the sequence starts at $n = 1$). It follows that

$$\mathcal{L}^i\left(\begin{bmatrix} n \\ 1 \end{bmatrix}\right) = (1, 0, 0, 0, \ldots)$$

for $i \geq 2$.

For $k = 2$, the manipulations are much like those in the previous paragraph. Using induction on i, we obtain

$$L^i\left(\begin{bmatrix} n \\ 2 \end{bmatrix}\right) = q^{(2^i-1)(n-2)}\begin{bmatrix} n \\ 2 \end{bmatrix}$$

for $i \geq 0$. This completes the proof of the last case of the proposition. $\square$

If we now consider arbitrary k it is not hard to show, using algebraic manipulations like those in the proof just given, that

$$L_n\left(\begin{bmatrix} n \\ k \end{bmatrix}\right) = \frac{q^{n-k}}{[n]}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} n \\ k-1 \end{bmatrix}. \tag{5}$$

These are, up to a power of q, the q-Narayana numbers. They were introduced by Fürlinger and Hofbauer [13] and are contained in a specialization of a result of MacMahon [20, page 1429] which was stated without proof. They were further studied by Brändén [5]. As shown in the references just cited, these polynomials are the generating functions for a number of different families of combinatorial objects. Thus they are q-nonnegative.

More computations show that

$$L_n^2\left(\begin{bmatrix} n \\ k \end{bmatrix}\right) = \frac{q^{3n-3k}[2]}{[n]^2[n-1]}\begin{bmatrix} n \\ k \end{bmatrix}^2\begin{bmatrix} n \\ k-1 \end{bmatrix}\begin{bmatrix} n \\ k-1 \end{bmatrix}. \tag{6}$$

It is not clear that these polynomials are q-nonnegative, although they must be if Conjecture 5.3 is true. Furthermore, when $q = 1$, the triangle made as n and k vary is not in Sloane's Encyclopedia [24] (although it has now been submitted). We expect that these integers and polynomials have interesting, yet to be discovered, properties.

We conclude our discussion of the Gaussian polynomials by considering the sequence

$$\left(\begin{bmatrix} n+mu \\ mv \end{bmatrix}\right)_{m\geq 0} \tag{7}$$

for nonnegative integers u and v, as we did in Section 4 for the binomial coefficients. When $u > v$ the sequence has an infinite number of nonzero entries. We can use (4) to show that the highest degree term in $\begin{bmatrix} n+u \\ v \end{bmatrix}^2 - \begin{bmatrix} n+2u \\ 2v \end{bmatrix}$ has coefficient -1, so the sequence (7) is not even q-log-concave. When $u < v$, it seems to be the case that the sequence is not 2-fold q-log-concave, as shown for the rows in Proposition 5.2. When $u = v$, the evidence is conflicting, reflecting the behavior of the binomial coefficients. Since setting $q = 1$ in $\begin{bmatrix} n+mu \\ mu \end{bmatrix}$ yields $\binom{n+mu}{mu}$, we know that $\left(\begin{bmatrix} 2+mu \\ mu \end{bmatrix}\right)_{m\geq 0}$ is not always 4-fold q-log-concave. It also transpires that the case $n = 3$ is not always 5-fold q-log-concave. We have not encountered other values of n that fail to yield a q-log-concave sequence when $u = v$.

While the variety of behavior of the Gaussian polynomials is interesting, it would be desirable to have a q-analogue that better reflects the behavior of the binomial coefficients. A q-analogue that arises in the study of quantum groups serves this purpose. Let us replace the previous q-analogue of the nonnegative integer n with

the expression

$$\langle n\rangle = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{1-n} + q^{3-n} + q^{5-n} + \cdots + q^{n-1}.$$

From this, we obtain a q-analogue of the binomial coefficients by proceeding as for the Gaussian polynomials: for $0 \le k \le n$, we define

$$\left\langle {n \atop k} \right\rangle = \frac{\langle n\rangle!}{\langle k\rangle!\langle n-k\rangle!}$$

where $\langle n\rangle! = \langle 1\rangle\langle 2\rangle \cdots \langle n\rangle$.

Letting $q \to 1$ in $\left\langle {n \atop k} \right\rangle$ gives $\binom{n}{k}$, and a straightforward calculation shows that

$$\left\langle {n \atop k} \right\rangle = \frac{1}{q^{nk-k^2}} \left[{n \atop k} \right]_{q^2}. \tag{8}$$

So $\left\langle {n \atop k} \right\rangle$ is, in general, a Laurent polynomial in q with nonnegative coefficients. Our definitions of q-nonnegativity and q-log-concavity for polynomials in q extend to Laurent polynomials in the obvious way.

We offer the following generalizations of Conjectures 1.1, 4.1 and 4.4.

Conjecture 5.5.

(a) The row sequence $\left(\left\langle {n \atop k} \right\rangle\right)_{k\ge 0}$ is infinitely q-log-concave for all $n \ge 0$.

(b) The column sequence $\left(\left\langle {n \atop k} \right\rangle\right)_{n\ge k}$ is infinitely q-log-concave for all fixed $k \ge 0$.

(c) For all integers $0 \le u < v$, the sequence $\left(\left\langle {n+mu \atop mv} \right\rangle\right)_{m\ge 0}$ is infinitely q-log-concave for all $n \ge 0$.

Several remarks are in order. Suppose that for $f(g), g(q) \in \mathbb{R}[q, q^{-1}]$, we say $f(q) \le g(q)$ if $g(q) - f(q)$ is q-nonnegative. Then the proofs of Lemmas 2.1 and 3.1 work equally well if the a_i's are Laurent polynomials and we replace the term "log-concave" by "q-log-concave." Using these lemmas, we have verified Conjecture 5.5(a) for all $n \le 53$. Even though (a) is a special case of (c), we state it separately since (a) is the q-generalization of the Boros-Moll conjecture, the primary motivation for this paper.

As evidence for Conjecture 5.5(b), it is not hard to prove the appropriate analogue of Propositions 4.2 and 5.4, i.e. that the sequence $\left\langle {n \atop k} \right\rangle_{n\ge k}$ is infinitely q-log-concave for all $0 \le k \le 2$. To obtain the expressions for $L_n\left(\left\langle {n \atop k} \right\rangle\right)$ and $L_n^2\left(\left\langle {n \atop k} \right\rangle\right)$, take equations (5) and (6), replace all square brackets by angle brackets and replace each the terms q^{n-k} and q^{3n-3k} by the number 1.

Conjecture 5.5(c) has been verified for all $n \leq 24$ with $v \leq 10$. When $u > v$, we can use (8) to show that the lowest degree term in $\left\langle {n+u \atop v} \right\rangle^2 - \left\langle {n+2u \atop 2v} \right\rangle$ has coefficient -1, so the sequence is not even q-log-concave. When $u = v$, the quantum groups analogue has exactly the same behavior as we observed above for the Gaussian polynomials.

6 Symmetric functions

We now turn our attention to symmetric functions. We will demonstrate that the complete homogeneous symmetric functions $(h_k)_{k\geq 0}$ are a natural analogue of the rows and columns of Pascal's triangle. We show that the sequence $(h_k)_{k\geq 0}$ is i-fold log-concave in the appropriate sense for $i \leq 3$, but not 4-fold log-concave. Like the results of Section 5, this result underlines the difficulties and subtleties of Conjectures 1.1 and 4.1. In particular, it shows that any proof of Conjecture 1.1 or Conjecture 4.1 would need to use techniques that do not carry over to the sequence $(h_k)_{k\geq 0}$. For a more detailed exposition of the background material below, we refer the reader to the texts of Fulton [12], Macdonald [19], Sagan [23] or Stanley [26].

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a countably infinite set of variables. For each $n \geq 0$, the elements of the symmetric group $\mathfrak{S}_n$ act on formal power series $f(\mathbf{x}) \in \mathbb{R}[[\mathbf{x}]]$ by permutation of variables (where x_i is left fixed if $i > n$). The algebra of symmetric functions, $\Lambda(\mathbf{x})$, is the set of all series left fixed by all symmetric groups and of bounded (total) degree.

The vector space of symmetric functions homogeneous of degree k has dimension equal to the number of partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of k. We will be interested in three bases for this vector space. The *monomial symmetric function* corresponding to λ, $m_\lambda = m_\lambda(\mathbf{x})$, is obtained by symmetrizing the monomial $x_1^{\lambda_1} \cdots x_\ell^{\lambda_\ell}$. The kth *complete homogeneous symmetric function*, h_k, is the sum of all monomials of degree k. For partitions, we then define

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}.$$

Finally, the *Schur function* corresponding to λ is

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1\leq i,j\leq \ell}.$$

We remark that this determinant is a minor of the Toeplitz matrix for the sequence (h_k). We will have more to say about Toeplitz matrices in the next section.

Our interest will be in the sequence just mentioned $(h_k)_{k\geq 0}$. Let $h_k(1^n)$ denote the integer obtained by substituting $x_1 = \cdots = x_n = 1$ and $x_i = 0$ for $i > n$ into $h_k = h_k(\mathbf{x})$. Then $h_k(1^n) = \binom{n+k-1}{k}$ (the number of ways of choosing k things from

n things with repetition) and so the above sequence becomes a column of Pascal's triangle. By the same token $h_k(1^{n-k}) = \binom{n-1}{k}$ and so the sequence becomes a row.

We will now collect the results from the theory of symmetric functions which we will need. Partially order partitions by *dominance* where $\lambda \leq \mu$ if and only if for every $i \geq 1$ we have $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$. Also, if $\{b_\lambda\}$ is any basis of $\Lambda(\mathbf{x})$ and $f \in \Lambda(\mathbf{x})$ then we let $[b_\lambda]f$ denote the coefficient of the basis element b_λ in the expansion of f in this basis. First we have a simple consequence of Young's Rule.

Theorem 6.1. *For any partitions λ, μ we have $[m_\mu]s_\lambda$ is a nonnegative integer. In particular,*

$$[m_\mu]s_\lambda = \begin{cases} 1 & \text{if } \mu = \lambda, \\ 0 & \text{if } \mu \not\leq \lambda. \end{cases} \qquad \square$$

Let $\lambda + \mu$ denote the componentwise sum $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$. The next result follows from the Littlewood-Richardson Rule and induction.

Theorem 6.2. *For any partitions $\lambda^1, \ldots, \lambda^r$ and μ we have $[s_\mu]s_{\lambda^1} \cdots s_{\lambda_r}$ is a nonnegative integer. In particular,*

$$[s_\mu]s_{\lambda^1} \cdots s_{\lambda^r} = \begin{cases} 1 & \text{if } \mu = \lambda^1 + \cdots + \lambda^r, \\ 0 & \text{if } \mu \not\leq \lambda^1 + \cdots + \lambda^r. \end{cases} \qquad \square$$

Because of this result we call $\lambda^1 + \cdots + \lambda^r$ the *dominant partition* for $s_{\lambda^1} \cdots s_{\lambda^r}$.

Finally, we need a result of Kirillov [17] about the product of Schur functions, which was proved bijectively by Kleber [18] and Fulmek and Kleber [11]. This result can be obtained by applying the Desnanot-Jacobi Identity—also known as Dodgson's condensation formula—to the Jacobi-Trudi matrix for $s_{k^{r+1}}$. Note that, to improve readability, we drop the sequence parentheses when a sequence appears as a subscript.

Theorem 6.3 ([11, 17, 18])**.** *For positive integers k, r we have*

$$(s_{k^r})^2 - s_{(k-1)^r} s_{(k+1)^r} = s_{k^{r-1}} s_{k^{r+1}}. \qquad \square$$

To state our results, we need a few more definitions. If b_λ is a basis for $\Lambda(\mathbf{x})$ and $f \in \Lambda(\mathbf{x})$ then we say f is b_λ-*nonnegative* if $[b_\lambda]f \geq 0$ for all partitions λ. Note that m_λ-nonnegativity is the natural generalization to many variables of the q-nonnegativity definition for $\mathbb{R}[q]$. Also note that s_λ-nonnegativity implies m_λ-nonnegativity by Theorem 6.1.

Theorem 6.4. *The sequence $\mathcal{L}^i(h_k)$ is s_λ-nonnegative for $0 \leq i \leq 3$. But the sequence $\mathcal{L}^4(h_k)$ is not m_λ-nonnegative.*

Proof. From the definition of the Schur function we have

$$L^0(h_k) = h_k = s_k \quad \text{and} \quad L^1(h_k) = (h_k)^2 - h_{k-1}h_{k+1} = s_{k^2}.$$

Now Theorem 6.3 immediately gives

$$L^2(h_k) = \left(s_{k^2}\right)^2 - s_{(k-1)^2}s_{(k+1)^2} = s_k s_{k^3}$$

which is s_λ-nonnegative by the first part of Theorem 6.2. Using Theorem 6.3 twice gives

$$\begin{aligned} L^3(h_k) &= \left(s_k\right)^2\left(s_{k^3}\right)^2 - s_{k-1}s_{(k-1)^3}s_{k+1}s_{(k+1)^3} \\ &= \left(s_k\right)^2\left(s_{k^3}\right)^2 - \left(s_k\right)^2 s_{(k-1)^3}s_{(k+1)^3} \\ &\quad + \left(s_k\right)^2 s_{(k-1)^3}s_{(k+1)^3} - s_{k-1}s_{(k-1)^3}s_{k+1}s_{(k+1)^3} \\ &= \left(s_k\right)^2 s_{k^2}s_{k^4} + s_{(k-1)^3}s_{k^2}s_{(k+1)^3} \end{aligned}$$

which is again s_λ-nonnegative. This finishes the cases $0 \le i \le 3$.

We now assume $k \ge 2$. Computing $L^4(h_k)$ from the expression for $L^3(h_k)$ gives the sum of the terms in the left column below. The right column gives the dominant partition for each term, as determined by Theorem 6.2.

$+(s_k)^4(s_{k^2})^2(s_{k^4})^2$	$(8k, 4k, 2k, 2k)$
$+2(s_k)^2(s_{k^2})^2 s_{k^4}s_{(k-1)^3}s_{(k+1)^3}$	$(7k, 5k, 3k, k)$
$+(s_{(k-1)^3})^2(s_{k^2})^2(s_{(k+1)^3})^2$	$(6k, 6k, 4k)$
$-(s_{k-1})^2 s_{(k-1)^2}s_{(k-1)^4}(s_{k+1})^2 s_{(k+1)^2}s_{(k+1)^4}$	$(8k, 4k, 2k, 2k)$
$-(s_{k-1})^2 s_{(k-1)^2}s_{(k-1)^4}s_{k^3}s_{(k+1)^2}s_{(k+2)^3}$	$(7k-1, 5k+1, 3k+1, k-1)$
$-s_{(k-2)^3}s_{(k-1)^2}s_{k^3}(s_{k+1})^2 s_{(k+1)^2}s_{(k+1)^4}$	$(7k+1, 5k-1, 3k-1, k+1)$
$-s_{(k-2)^3}s_{(k-1)^2}(s_{k^3})^2 s_{(k+1)^2}s_{(k+2)^3}$	$(6k, 6k, 4k)$

Now consider $\lambda = (7k+1, 5k-1, 3k-1, k+1)$, the dominant partition for the penultimate term above. Observe that if μ is the dominant partition for any other term, then $\lambda \not\le \mu$. So, by the second part of Theorem 6.2, s_λ appears in the Schur-basis expansion for $L^4(h_k)$ with coefficient -1. It then follows from the second part of Theorem 6.1, that the coefficient of m_λ is -1 as well. □

7 Real roots and Toeplitz matrices

We now consider two other (almost equivalent) settings where, in contrast to the results of the previous section, Conjecture 1.1 does seem to generalize. In fact, this may be the right level of generality to find a proof.

Let $(a_k) = a_0, a_1, \ldots, a_n$ be a finite sequence of nonnegative real numbers. It was shown by Isaac Newton that if all the roots of the polynomial $p[a_k] \stackrel{\text{def}}{=} a_0 + a_1 x + \cdots a_n x^n$ are real, then the sequence (a_k) is log-concave. For example, since the polynomial $(1+x)^n$ has only real roots, the nth row of Pascal's triangle is log-concave. It is natural to ask if the real-rootedness property is preserved by the $\mathcal{L}$-operator. The literature includes a number of results about operations on polynomials which preserve real-rootedness; for example, see [6, 7, 8, 22, 31, 32].

Conjecture 7.1. *Let (a_k) be a finite sequence of nonnegative real numbers. If $p[a_k]$ has only real roots then the same is true of $p[\mathcal{L}(a_k)]$.*

This conjecture is due independently to Richard Stanley [private communication]. It is also one of a number of related conjectures made by Steve Fisk [10]. If true, Conjecture 7.1 would immediately imply the original Boros-Moll Conjecture. As evidence for the conjecture, we have verified it by computer for a large number of randomly chosen real-rooted polynomials. We have also checked that $p[\mathcal{L}_k^i \binom{n}{k}]$ has only real roots for all $i \le 10$ and $n \le 40$. It is interesting to note that Boros and Moll's polynomial $P_m(x)$ in equation (2) does not have real roots even for $m = 2$. So if the corresponding sequence is infinitely log-concave then it must be so for some other reason.

Along with the rows of Pascal's triangle, it appears that applying $\mathcal{L}$ to the other finite lines we were considering in Section 4 also yields sequences with real-rooted generating functions. So we make the following conjecture which implies the "if" direction of Conjecture 4.4.

Conjecture 7.2. *For $0 \le u < v$, the polynomial $p[\mathcal{L}_m^i(\binom{n+mu}{mv})]$ has only real roots for all $i \ge 0$.*

We have verified this assertion for all $n \le 24$ with $i \le 10$ and $v \le 10$. In fact, it follows from a theorem of Yu [33] that the conjecture holds for $i = 0$ and all $0 \le u < v$. So it will suffice to prove Conjecture 7.1 to obtain this result for all i.

We can obtain a matrix-theoretic perspective on problems of real-rootedness via the following renowned result of Aissen, Schoenberg and Whitney [1]. A matrix A is said to be totally nonnegative if every minor of A is nonnegative. We can associate with any sequence (a_k) a corresponding (infinite) *Toeplitz matrix* $A = (a_{j-i})_{i,j\ge 0}$. In comparing the next theorem to Newton's result, note that for a real-rooted polynomial $p[a_k]$ the roots being nonpositive is equivalent to the sequence (a_k) being nonnegative.

Theorem 7.3 ([1])**.** *Let (a_k) be a finite sequence of real numbers. Then every root of $p[a_k]$ is a nonpositive real number if and only if the Toeplitz matrix $(a_{j-i})_{i,j\ge 0}$ is totally nonnegative.* □

To make a connection with the $\mathcal{L}$-operator, note that

$$a_k^2 - a_{k-1}a_{k+1} = \begin{vmatrix} a_k & a_{k+1} \\ a_{k-1} & a_k \end{vmatrix},$$

which is a minor of the Toeplitz matrix $A = (a_{j-i})_{i,j\geq 0}$. Call such a minor *adjacent* since its entries are adjacent in A. Now, for an arbitrary infinite matrix $A = (a_{i,j})_{i,j\geq 0}$, let us define the infinite matrix $\mathcal{L}(A)$ by

$$\mathcal{L}(A) = \left(\begin{vmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} \right)_{i,j\geq 0}.$$

Note that if A is the Toeplitz matrix of (a_k) then $\mathcal{L}(A)$ is the Toeplitz matrix of $\mathcal{L}(a_k)$. Using Theorem 7.3, Conjecture 7.1 can now be strengthened as follows.

Conjecture 7.4. *For a sequence (a_k) of real numbers, if $A = (a_{j-i})_{i,j\geq 0}$ is totally nonnegative then $\mathcal{L}(A)$ is also totally nonnegative.*

Note that if (a_k) is finite, then Conjecture 7.4 is equivalent to Conjecture 7.1. As regards evidence for Conjecture 7.4, consider an arbitrary n-by-n matrix $A = (a_{i,j})_{i,j=1}^n$. For finite matrices, $\mathcal{L}(A)$ is defined in the obvious way to be the $(n-1)$-by-$(n-1)$ matrix consisting of the 2-by-2 adjacent minors of A. In [9, Theorem 6.5], Fallat, Herman, Gekhtman, and Johnson show that for $n \leq 4$, $\mathcal{L}(A)$ is totally nonnegative whenever A is. However, for $n = 5$, an example from their paper can be modified to show that if

$$A = \begin{pmatrix} 1 & t & 0 & 0 & 0 \\ t & t^2+1 & 2t & t^2 & 0 \\ t^2 & t^3+2t & 1+4t^2 & 2t^3+t & 0 \\ 0 & t^2 & 2t^3+2t & t^4+2t^2+1 & t \\ 0 & 0 & t^2 & t^3+t & t^2 \end{pmatrix}$$

then A is totally nonnegative for $t \geq 0$, but $\mathcal{L}(A)$ is not totally nonnegative for sufficiently large t ($t \geq \sqrt{2}$ will suffice). We conclude that the Toeplitz structure would be important to any affirmative answer to Conjecture 7.4.

We finish our discussion of the matrix-theoretic perspective with a positive result similar in flavor to Conjecture 7.4.

Proposition 7.5. *If A is a finite square matrix that is positive semidefinite, then $\mathcal{L}(A)$ is also positive semidefinite.*

Proof. The key idea is to construct the *second compound matrix* $\mathcal{C}_2(A)$ of A, which is the array of all 2-by-2 minors of A, arranged lexicographically according to the row and column indices of the minors [14].

We claim that if A is positive semidefinite, then so is $\mathcal{C}_2(A)$. Indeed, since the compound operation preserves multiplication and inverses, the eigenvalues of $\mathcal{C}_2(A)$ are equal to the eigenvalues of $\mathcal{C}_2(J)$, where J is the Jordan form of A. If J is upper-triangular and has diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$, then we see that $\mathcal{C}_2(J)$ is upper-triangular with diagonal entries $\lambda_i \lambda_j$ for all $i < j$. Since the λ_i's are all nonnegative, so too are the eigenvalues of $\mathcal{C}_2(J)$, implying that $\mathcal{C}_2(A)$ is positive semidefinite.

Finally, since $\mathcal{L}(A)$ is a principal submatrix of $\mathcal{C}_2(A)$, $\mathcal{L}(A)$ is itself positive semidefinite. $\square$

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